

Bound on remote preparation of entanglement from isotropic states

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Using the negativity as an entanglement measure, we investigate the possible amount of remotely prepared entanglement. For two identical isotropic states on two-qudit systems 12 and 34, we calculate the average amount of entanglement remotely distributed on the system 13 by joint measurement on the system 24, and show that the remote preparation of entanglement by the generalized Bell-measurement is optimal among rank-one measurements if the isotropic states have a certain fidelity with a maximally entangled state in higher dimensional quantum systems, or if the fidelity of the isotropic states is greater than a certain value depending on the dimension. In addition, we construct a measurement better than the generalized Bell-measurement with respect to the remote preparation of entanglement when the isotropic states have small fidelity.

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I. INTRODUCTION

Entanglement provides us with a novel correlation between two or more parties, which cannot be explained by any classical theories. In addition, the correlation can be successfully applied to quantum communication, which seems to be classically impossible. Thus entanglement shared between several parties has been considered as one of the most significant resources in quantum information processing including quantum teleportation [1] and quantum key distribution [2–4].

In order to perform a faithful quantum communication procedure in a given quantum network, first of all, it should be required to prepare sufficient entanglement shared between desired parties. As a generalization of entanglement swapping [5–7], there exists a process to remotely distribute entanglement between different parties [8–11], which is here called the *remote preparation of entanglement* (RPE).

One of the simplest cases of the RPE is as follows (see FIG. 1): Assume that Alice (system 2), Bob (system 4) and the supplier Sapna (systems 1 and 3) share an initial state $\rho^{12} \otimes \rho^{34} \in \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \mathcal{H}_3 \otimes \mathcal{H}_4)$, and Sapna performs a joint measurement on systems 1 and 3. Then entanglement can be probabilistically shared between Alice and Bob.

It has been known [8–11] that the possible average amount of the shared entanglement is bounded by the product of entanglement amounts of ρ^{12} and ρ^{34} . In particular, if ρ^{12} and ρ^{34} are two-qubit states then the bound can be shown in terms of the *concurrence* \mathcal{C} [12] as follows [8]:

$$\sum_{j=1}^s Q_j \mathcal{C}(\sigma_j^{24}) \leq \mathcal{C}(\rho^{12}) \mathcal{C}(\rho^{34}), \quad (1)$$

where Q_j is the probability of the resulting state σ_j^{24} on

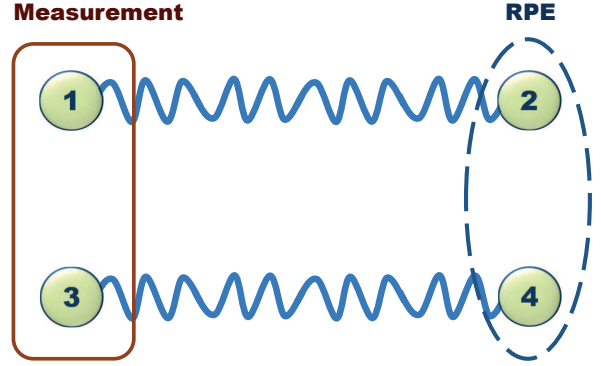


FIG. 1: A simple example of remote preparation of entanglement.

subsystems 24 after the joint measurement on subsystems 13, and $\mathcal{C}(\rho^{12})$, $\mathcal{C}(\rho^{34})$ and $\mathcal{C}(\sigma_j^{24})$ are the concurrence of ρ^{12} , ρ^{34} and σ_j^{24} , respectively.

However, we remark that the maximal average amount of entanglement by the RPE cannot generally attain to the bound, the product of entanglement amounts of the two states. Especially, for two-qubit states ρ^{12} and ρ^{34} , there does not in general exist a measurement to saturate the inequality (1).

We note that if we can discover a measurement to achieve the maximal average amount of entanglement by the RPE in a given quantum network then we can learn how faithful quantum communication can be performed in the quantum network, and this can be also employed in constituting a quantum network for a desired level of quantum communication. On this account, it may be more important to calculate the maximal average amount of entanglement obtainable by the RPE or to find out the optimal measurement for the RPE than to acquire a constant bound in a given quantum network.

In order to evaluate amount of entanglement, we need an entanglement quantification. Since the negativity is

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one of the most computable measures of entanglement among several entanglement quantifications, we here deal with the *negativity* \mathcal{N} [13–15], which is defined as

$$\mathcal{N}(\rho) \equiv \frac{\|\rho^\Gamma\| - 1}{d - 1} \quad (2)$$

for a two-qudit state ρ , where $\|\cdot\|$ is the trace norm, and Γ is the partial transposition. Moreover, since the negativity is an entanglement monotone [13–15], it can be regarded as a good measure of entanglement, even though the bound entanglement with positive partial transposition [16, 17] cannot be detected by the negativity.

In this paper, we investigate the possible amount of entanglement by the RPE in terms of the negativity, and show that, given two identical isotropic states as initial states, the RPE by the generalized Bell-measurement is optimal among rank-one measurements if the isotropic states have a certain fidelity with a maximally entangled state in higher dimensional quantum systems, or if the fidelity of the isotropic states is greater than a certain value depending on the dimension of quantum systems. In addition, we show that there exists a measurement better than the generalized Bell-measurement with respect to the RPE when the isotropic states have small fidelity.

II. REMOTE PREPARATION OF ENTANGLEMENT AND THE GENERALIZED BELL-MEASUREMENT

In this section, we show that the generalized Bell-measurement among rank-one measurements is an optimal measurement for the RPE presented in FIG. 1, provided that two identical isotropic states with a certain fidelity are initial states.

Let

$$\begin{aligned} \rho_F &= F|\phi_d\rangle\langle\phi_d| + \frac{1-F}{d^2-1}(I \otimes I - |\phi_d\rangle\langle\phi_d|) \\ &= F\Phi_d + \frac{1-F}{d^2-1}(\mathcal{I} - \Phi_d) \end{aligned} \quad (3)$$

be the isotropic state with fidelity F in the two-qudit system, where $|\phi_d\rangle = \sum_{j=0}^{d-1} |jj\rangle/\sqrt{d}$ is a maximally entangled state in the two-qudit system, I is the d -dimensional identity operator, $\Phi_d = |\phi_d\rangle\langle\phi_d|$ and $\mathcal{I} = I \otimes I$. We note that the isotropic states form a generic class of two-qudit states, to which any arbitrary two-qudit state can be transformed via local quantum operations and classical communication, and moreover it is not difficult to analyze their amounts of entanglement computed by the negativity. Hence quantum networks consisting of the isotropic states are here dealt with.

We consider $\rho_F^{12} \otimes \rho_F^{34}$ as an initial state. For the sake of convenience, we let $a = (1-F)/(d^2-1)$ and $b = (d^2F-1)/(d^2-1)$. Then it is clear that $d^2a+b=1$, and

$\rho_F^{12} \otimes \rho_F^{34}$ becomes

$$\begin{aligned} \varrho &\equiv \rho_F^{12} \otimes \rho_F^{34} \\ &= (a\mathcal{I}^{12} + b\Phi_d^{12}) \otimes (a\mathcal{I}^{34} + b\Phi_d^{34}). \end{aligned} \quad (4)$$

We now suppose that a rank-one measurement on subsystems 13 is performed, and let $|\psi\rangle\langle\psi|$ be a measurement operator of the rank-one measurement. Due to the symmetry of isotropic states over local unitary operations, without loss of generality, we may assume that

$$|\psi\rangle = \sum_{j=0}^{d-1} \sqrt{\lambda_j} |jj\rangle = \sum_{j=0}^{R-1} \sqrt{\lambda_j} |jj\rangle, \quad (5)$$

where $1 \geq \lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{R-1} > 0$, and $\lambda_R = \lambda_{R+1} = \dots = \lambda_{d-1} = 0$. It follows from straightforward calculations that

$$\begin{aligned} {}^{13}\langle\psi|\varrho|\psi\rangle^{13} &= a^2\mathcal{I}^{24} + \frac{b^2}{d^2} \sum_{i,j=0}^{R-1} \sqrt{\lambda_i\lambda_j} |ii\rangle^{24} \langle jj| \\ &\quad + \frac{ab}{d} (\Lambda_R \otimes I + I \otimes \Lambda_R)^{24}, \end{aligned} \quad (6)$$

where

$$\Lambda_R = \sum_{j=0}^{R-1} \lambda_j |j\rangle\langle j|. \quad (7)$$

Then $\text{tr}\langle\psi|\varrho|\psi\rangle = d^2a^2 + b^2/d^2 + 2ab = 1/d^2$. Thus, the probability to obtain one measurement outcome after the rank-one measurement is $1/d^2$, and the resulting state ρ_ψ in subsystems 24 after the measurement becomes

$$\begin{aligned} \rho_\psi &\equiv d^2 \cdot {}^{13}\langle\psi|\varrho|\psi\rangle^{13} \\ &= d^2a^2\mathcal{I}^{24} + b^2 \sum_{i,j=0}^{R-1} \sqrt{\lambda_i\lambda_j} |ii\rangle^{24} \langle jj| \\ &\quad + dab(\Lambda_R \otimes I + I \otimes \Lambda_R)^{24}. \end{aligned} \quad (8)$$

We note that $A \otimes B|\phi_d\rangle = I \otimes BA^T|\phi_d\rangle$ for any operators A and B on the d -dimensional quantum system. Hence, it can be clearly shown that one outcome after performing an arbitrary rank-one measurement on subsystems 13 is always obtained with probability $1/d^2$, and it can be also shown that the resultant state in subsystems 24 is equivalent to the state ρ_ψ up to local unitary operations, if the pure state corresponding to the measurement outcome has the same Schmidt coefficients as those of the state $|\psi\rangle$.

We now take into account the partial transpose on subsystem 4. Then the partial transpose of ρ_ψ is

$$\begin{aligned} \rho_\psi^\Gamma &= d^2a^2\mathcal{I}^{24} + b^2 \sum_{i,j=0}^{R-1} \sqrt{\lambda_i\lambda_j} |ij\rangle^{24} \langle ji| \\ &\quad + dab(\Lambda_R \otimes I + I \otimes \Lambda_R)^{24}. \end{aligned} \quad (9)$$

Thus, for each $0 \leq k \leq d-1$, we have

$$\rho_\psi^\Gamma |kk\rangle = (d^2 a^2 + (b^2 + 2dab)\lambda_k) |kk\rangle. \quad (10)$$

For each $0 \leq k < l \leq d-1$, let $|\psi_{kl}^\pm\rangle = (|kl\rangle \pm |lk\rangle)/\sqrt{2}$, then we also have

$$\rho_\psi^\Gamma |\psi_{kl}^\pm\rangle = \left(d^2 a^2 + dab(\lambda_k + \lambda_l) \pm b^2 \sqrt{\lambda_k \lambda_l} \right) |\psi_{kl}^\pm\rangle. \quad (11)$$

It follows from Eqs. (10) and (11) that $|kk\rangle$'s and $|\psi_{kl}^\pm\rangle$'s are eigenvectors of ρ_ψ , and form an orthonormal basis for the two-qudit system. Hence, we can obtain that the negativity of ρ_ψ , $\mathcal{N}(\rho_\psi)$, is

$$\frac{2}{d-1} \sum_{k < l < R} \max \left\{ 0, b^2 \sqrt{\lambda_k \lambda_l} - d^2 a^2 - dab(\lambda_k + \lambda_l) \right\}, \quad (12)$$

which is less than or equal to

$$\frac{2}{d-1} \sum_{k < l < R} \max \left\{ 0, (b^2 - 2dab) \sqrt{\lambda_k \lambda_l} - d^2 a^2 \right\}. \quad (13)$$

It is clear that Eq. (13) becomes

$$\frac{2}{d-1} \sum_{(k,l) \in \mathcal{J}_\psi} \left((b^2 - 2dab) \sqrt{\lambda_k \lambda_l} - d^2 a^2 \right), \quad (14)$$

where \mathcal{J}_ψ is the set of all ordered pairs (k, l) such that $k < l < R$ and $(b^2 - 2dab) \sqrt{\lambda_k \lambda_l} \geq d^2 a^2$.

Since $a = (1 - F)/(d^2 - 1)$ and $b = (d^2 F - 1)/(d^2 - 1)$, it can be shown that if d is sufficiently large to satisfy

$$b^2 - 2dab \geq 2(R-1)d^2 a^2 \quad (15)$$

then

$$\begin{aligned} \mathcal{N}(\rho_\psi) &\leq \frac{R-1}{d-1} (b^2 - 2dab - Rd^2 a^2) \\ &= (b^2 - 2dab) \mathcal{N}(\Phi_R) - \frac{R(R-1)}{d-1} d^2 a^2 \\ &= \mathcal{N}(\rho_{\phi_R}). \end{aligned} \quad (16)$$

Furthermore, it can be also shown that if the dimension d is large enough to satisfy

$$b^2 - 2dab \geq (R+d-1)d^2 a^2 \quad (17)$$

then

$$\frac{R-1}{d-1} (b^2 - 2dab - Rd^2 a^2) \quad (18)$$

is less than or equal to

$$b^2 - 2dab - d^3 a^2, \quad (19)$$

that is, the inequality $\mathcal{N}(\rho_{\phi_R}) \leq \mathcal{N}(\rho_{\phi_d})$ holds. Thus, we obtain that if the dimension d is sufficiently large such that

$$b^2 - 2dab \geq 2(d-1)d^2 a^2 \quad (20)$$

then

$$\mathcal{N}(\rho_\psi) \leq \mathcal{N}(\rho_{\phi_d}) \quad (21)$$

for any state $|\psi\rangle$. Hence, it can be obtained that, given a rank-one measurement of subsystem 13 with measurement operators $\{|\psi_j\rangle\langle\psi_j|\}$ on the state ϱ , if the inequality (20) holds then

$$\sum_j p(\psi_j) \mathcal{N}(\rho_{\psi_j}) \leq \sum_{s,t} p(\phi_{st}) \mathcal{N}(\rho_{\phi_{st}}), \quad (22)$$

where ρ_{ψ_j} 's are the resultant states on subsystem 24 after the rank-one measurement, $p(\zeta)$ is the probability to obtain ζ as a measurement outcome, and $|\phi_{st}\rangle = I \otimes X^s Z^t |\phi_d\rangle$ with the generalized Pauli operators X and Z are the two-qudit generalized Bell states, since $p(\psi_j) = p(\phi_{st}) = 1/d^2$ and $\mathcal{N}(\rho_{\psi_j}) \leq \mathcal{N}(\rho_{\phi_d}) = \mathcal{N}(\rho_{\phi_{st}})$.

We remark that the left-hand side and the right-hand side in the inequality (22) represent the average amount of entanglement of the resultant states after the rank-one measurement and after the generalized Bell measurement, respectively. This implies that, under the condition in the inequality (20), the generalized Bell measurement is optimal among rank-one measurements with respect to the RPE.

In addition, the inequality (22) also holds if F is more than a certain value depending on the dimension so that

$$F \geq \frac{1 + 3d - d^2 + (d^2 - 1)\sqrt{2d-1}}{d(d^2 + 2)}, \quad (23)$$

since the inequality (23) is equivalent to the inequality (20). Therefore, we can obtain the following theorem.

Theorem 1. *Assume that the initial states are two identical isotropic states in the two-qudit quantum system. Then the RPE by the generalized Bell-measurement is optimal among rank-one measurements if the dimension of the quantum system is large enough to satisfy the inequality (23), or if the isotropic states have fidelity more than a certain value depending on the dimension as seen in the inequality (23).*

We remark that for almost all initial states in higher dimensional quantum systems, the RPE by the generalized Bell-measurement is optimal, since the right-hand side in the inequality (23) tends to zero as the dimension d goes to the infinity.

III. MEASUREMENTS BETTER THAN THE GENERALIZED BELL-MEASUREMENT

In this section, we show that there exists a measurement to give higher entanglement than the generalized Bell-measurement for the RPE when the two identical isotropic states as initial states have small fidelity.

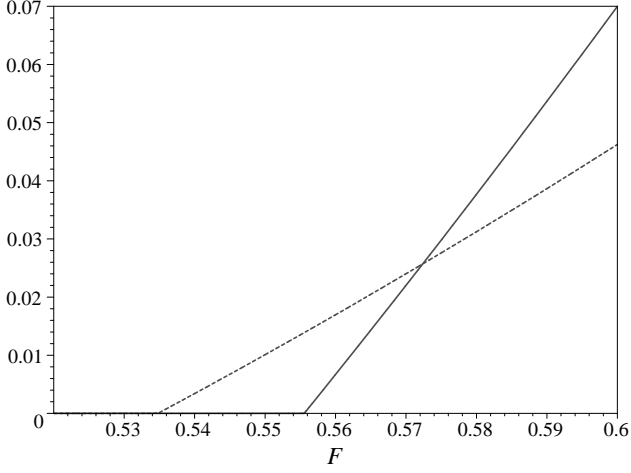


FIG. 2: The values of $\mathcal{N}(\rho_{\phi_3})$ and $\mathcal{N}(\rho_{\psi_{kl}^\pm})$ in the qutrit systems: The solid and dotted curves represent the values of $\mathcal{N}(\rho_{\phi_3})$ and $\mathcal{N}(\rho_{\psi_{kl}^\pm})$, respectively: $\mathcal{N}(\rho_{\psi_{kl}^\pm})$ is more than $\mathcal{N}(\rho_{\phi_3})$ for $0.535 \approx (7 + 8\sqrt{3})/39 < F < (1 + 8\sqrt{5})/33 \approx 0.572$.

A. Three-dimensional quantum systems

In this subsection, we assume that $d = 3$. Let $\omega_3 = \exp(2\pi i/3)$ with $i = \sqrt{-1}$, and consider a measurement whose measurement operators are $|\psi_{kl}^\pm\rangle\langle\psi_{kl}^\pm|$ for $0 \leq k < l \leq 2$ and $|\phi_3^s\rangle\langle\phi_3^s|$ for $0 \leq s \leq 2$, where

$$|\phi_3^s\rangle \equiv \sum_{j=0}^2 \omega_3^{sj} |jj\rangle. \quad (24)$$

Then it can be easily shown that the measurement is a rank-one projective measurement, and it can be also shown that $\mathcal{N}(\rho_{\psi_{kl}^\pm})$ is more than $\mathcal{N}(\rho_{\phi_3})$ for

$$\frac{7 + 8\sqrt{3}}{39} < F < \frac{1 + 8\sqrt{5}}{33}, \quad (25)$$

as seen in FIG. 2.

It follows that if the given isotropic states have fidelity F satisfying the inequality (25) then the RPE by this measurement provides higher entanglement than the RPE by the generalized Bell-measurement in terms of the negativity.

B. d -dimensional quantum systems

In this subsection, we generalize the example in the three-dimensional case. Let $\omega_d = \exp(2\pi i/d)$, and con-

sider a measurement whose measurement operators are $|\psi_{kl}^\pm\rangle\langle\psi_{kl}^\pm|$ for $0 \leq k < l \leq d-1$ and $|\phi_d^s\rangle\langle\phi_d^s|$ for $0 \leq s \leq d-1$, where

$$|\phi_d^s\rangle \equiv \sum_{j=0}^{d-1} \omega_d^{sj} |jj\rangle. \quad (26)$$

Then, as in the three-dimensional case, the measurement is also a rank-one projective measurement.

It follows from tedious but straightforward calculations that if the fidelity F of the initial isotropic states satisfies the following inequality

$$\frac{1 - d + d^2 + (d^2 - 1)\sqrt{3}}{d(d^2 + 2d - 2)} < F < \frac{1 + (d^2 - 1)\sqrt{d+2}}{d(d^2 + d - 1)}, \quad (27)$$

then the measurement gives higher amount of entanglement than the generalized Bell-measurement for the RPE.

From the two previous subsections, we have the following theorem.

Theorem 2. *There exists a measurement such that the RPE by the measurement has larger average amount of entanglement than the RPE by the generalized Bell-measurement.*

IV. SUMMARY

We have investigated the possible amount of entanglement by the RPE in terms of the negativity, and have shown that, given two identical isotropic states as initial states, the RPE by the generalized Bell-measurement is optimal among rank-one measurements if the isotropic states have a certain fidelity with a maximally entangled state in higher dimensional quantum systems, or if the fidelity of the isotropic states is greater than a certain value depending on the dimension of the quantum systems.

On the other hand, we have shown that there exists a measurement better than the generalized Bell-measurement with respect to the RPE when the isotropic states have small fidelity.

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